Analysis of a Radial Crack in Cross-Ply Laminates Under Uniaxial Tension

Seong-Kyun Cheong* and Oh-Nam Kwon**

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The problem of a radial crack in cross-ply laminates under uniaxial tension is investigated in this paper. The normalized stress intensity factors are obtained by the modified mapping -collocation method which is based on analytic complex function theory of complex variables. The present results for an isotropic infinite plate show good agreement with existing solutions. In the range of small crack length, the stress intensity factor for a radial crack in cross-ply laminates under uniaxial tension becomes larger as the percentage of 0° plies increases. However in the range of large crack length, it is insensitive to the percentage of 0° plies.

Key Words: Modified Mapping-Collocation Method, Cross-Ply Laminates, Radial Crack, Stress Intensity Factor, Analytic Complex Function

1. Introduction

Cracks can be found around areas of stress concentrations under cyclic loads. Holes are good examples for stress concentrations in structures. Therefore, the problem of a radial crack needs to be studied.

Many papers (Bowie, 1956; Hsu, 1975; Shivakumar and Forman, 1980; Newman Jr., 1971; Tweed and Rooke, 1973; Tweed and Rooke, 1976), which investigated the stress intensity factors for cracks around areas of stress concentrations in isotropic plates, were published. The stress intensity factors were obtained by various techniques in these papers.

However, only a few papers (Waddoups, Eisenmann, and Kaminski, 1971; Wang and Yau, 1980) deal with anisotropic materials can be found due to the complexity of material anisotropy and geometry. Waddoups et al. (1971) applied Bowie's solution for an isotropic infinite plate to the analysis of the finite-dimensional anisotropic composite. Wang and Yau (1980) used the path-indep endent J-integral for an arbitrary path to solve the problem.

Recently, authors (Cheong and Hong, 1988; Cheong and Hong, 1989; Cheong and Kwon, 1993) investigated the cracks around areas of stress concentrations in laminated composites by using a modified mapping-collocation method. The geometric shapes of cracks in these papers are assumed to be symmetric. It is difficult to solve a problem of a single crack emanating from a circular hole in laminates because of geometric asymmetry.

In this study, the correction factors for a radial crack in cross-ply laminates under uniaxial tension is calculated by the modified mapping collocation method which is based on analytic complex function theory of complex variables. The present results for the case of is otropic infinite plate under uniaxial tension are compared with those of references (Bowie, 1956; Tweed and Rooke, 1973). Then, numerical calculations are performed for a single crack emanating from a circular hole in various types of cross-ply laminates under uniaxial tension.

* Department of Mechanical Engineering, Seoul National Polytechnic University, 172 Gongneung -dong, Nowon-gu, Seoul 139-743, Korea

** Department of Mathematics Education, Ewha Womans University, 11-1 Daehyun-dong, Seodaemun-gu, Seoul 120-750, Korea

2. Basic Equation of Two-Dimensional Anisotropic Elasticity

When body forces are absent or are constant, the differential equations of equilibrium are

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0$$
$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \qquad (1)$$

The equation of compatibility is

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}-\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}=0. \qquad (2)$$

The stress-strain relations for an anisotropic material in plane stress can be expressed as follows:

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{16}\\ a_{12} & a_{22} & a_{26}\\ a_{16} & a_{26} & a_{66}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} \qquad (3)$$

where $a_{ij}s$ are compliance components.

The differential equations of equilibrium are satisfied by the introduction of a stress function $F(x, y)$ and by assuming that

$$\sigma_x=\frac{\partial^2 F}{\partial y^2},\ \sigma_y=\frac{\partial^2 F}{\partial x^2},\ \tau_{xy}=-\frac{\partial^2 F}{\partial x \partial y} \qquad (4)$$

From Eqs (2), (3), and (4), the general form of the stress function can be obtained as (Lekhnitskii, 1968)

$$F(x, y)=2Re[F_1(z_1)+F_2(z_2)] \qquad (5)$$

where

$$z_k=x+s_k y \quad (k=1, 2) \qquad (6)$$

and F_1 and F_2 are analytic functions of the complex variables z_1 and z_2, respectively. The complex parameters s_1 and s_2 are roots of characteristic equation given as (Lekhnitskii, 1968)

$$a_{11}s^4-2a_{16}s^3+(2a_{12}+a_{66})s^2-2a_{26}s+a_{22}=0 \qquad (7)$$

Substituting Eq. (5) into Eq. (4), the stress components are

$$\sigma_x=2Re[s_1^2\phi_1'(z_1)+s_2^2\phi_2'(z_2)]$$
$$\sigma_y=2Re[\phi_1'(z_1)+\phi_2'(z_2)]$$
$$\tau_{xy}=-2Re[s_1\phi_1'(z_1)+s_2\phi_2'(z_2)] \qquad (8)$$

where

$$\phi_k(z_k)=F_k'(z_k) \quad (k=1, 2) \qquad (9)$$

From Eq. (3) and the strain-displacement relations, a simple integration gives the displacement components u and v :

$$u=2Re[p_1\phi_1(z_1)+p_2\phi_2(z_2)]$$
$$v=2Re[q_1\phi_1(z_1)+q_2\phi_2(z_2)] \qquad (10)$$

where p_k, $q_k (k=1, 2)$ are defined by

$$p_k=a_{11}s_k^2+a_{12}-a_{16}s_k$$
$$q_k=(a_{12}s_k^2+a_{22}-a_{26}s_k)/s_k \quad (k=1, 2) \qquad (11)$$

The boundary conditions of the traction type may also be expressed as

$$f_1(s)+if_2(s)=i\int^s (X_n+iY_n)\,ds$$
$$=(1+is_1)\phi_1(z_1)+(1+is_2)\phi_2(z_2)$$
$$+\overline{(1+is_1)}\,\overline{\phi_1(z_1)}+\overline{(1+is_2)}\,\overline{\phi_2(z_2)}+c \qquad (12)$$

where X_n and Y_n are the x and y components of forces exerted upon the edge per unit area. The bar notation is a conjugate symbol.

3. Theoretical Developments

We consider a radial crack in cross-ply laminates as shown in Fig. 1. Because of the complex crack geometry and strong material orthotropy, it is not so easy to solve this problem. The modified mapping-collocation method will be used, which was introduced in Bowie and Neal (1970), and Bowie and Freese (1972).

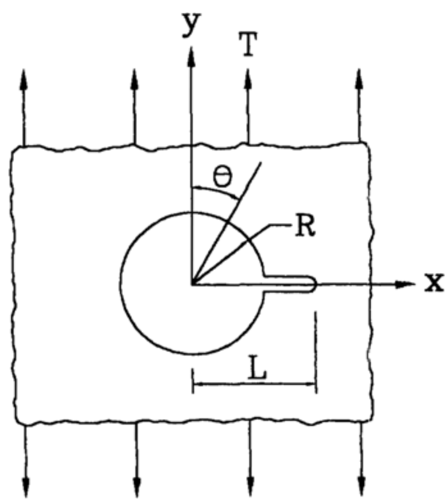

Fig. 1 Radial crack in cross-ply laminates under uniaxial tension

We introduce the transformation

$$z=\omega(\zeta)=\frac{L}{4}\left(\zeta+\frac{1}{\zeta}\right)+\frac{L}{2} \qquad (13)$$

The above mapping function carries the unit circle and its exterior in the ζ-plane into the crack and its exterior. The other boundaries correspond to a closed contour in the ζ-plane exterior to the unit circle with co-ordinate points

$$\zeta=\frac{2}{L}\left[\left(z-\frac{L}{2}\right)+z^{\frac{1}{2}}(z-L)^{\frac{1}{2}}\right] \qquad (14)$$

We consider now the complex variables z_1, z_2 and the additional relations:

$$\begin{aligned} z_k &= \omega(\zeta_k) \\ &= \frac{L}{4}\left(\zeta_k+\frac{1}{\zeta_k}\right)+\frac{L}{2} \quad (k=1,\ 2) \end{aligned} \qquad (15)$$

Since $z=z_1=z_2$ on the crack, the parameter planes ζ, ζ_1, and ζ_2 coincide on the unit circle. Otherwise, ζ_1 and ζ_2 are distinct and are found from

$$\zeta_k=\frac{2}{L}\left[\left(z_k-\frac{L}{2}\right)+z_k^{\frac{1}{2}}(z_k-L)^{\frac{1}{2}}\right]\cdot \quad (k=1,\ 2) \qquad (16)$$

For convenience, we now define the following useful notation :

$$\begin{aligned} &\phi_k(z_k)=\phi[\omega(\zeta_k)]=\phi_k(\zeta_k), \\ &\phi_k'(z_k)=\phi_k'(\zeta_k)/\omega'(\zeta_k) \quad (k=1,\ 2) \end{aligned} \qquad (17)$$

where

$$\omega(\zeta_k)=\frac{L}{4}\left(1-\frac{1}{\zeta_k^2}\right) \quad (k=1,\ 2) \qquad (18)$$

From Eqs. (8) and (17), the stresses in terms of $\phi_1(\zeta_1)$ and $\phi_2(\zeta_2)$ are

$$\begin{aligned} \sigma_x &= 2Re\left[s_1^2\frac{\phi_1'(\zeta_1)}{\omega'(\zeta_1)}+s_2^2\frac{\phi_2'(\zeta_2)}{\omega'(\zeta_2)}\right] \\ \sigma_y &= 2Re\left[\frac{\phi_1'(\zeta_1)}{\omega'(\zeta_1)}+\frac{\phi_2'(z_2)}{\omega'(\zeta_2)}\right] \\ \tau_{xy} &= -2Re\left[s_1\frac{\phi_1'(\zeta_1)}{\omega'(\zeta_1)}+s_2\frac{\phi_2'(\zeta_2)}{\omega'(\zeta_2)}\right] \end{aligned} \qquad (19)$$

From Eqs. (10) and (17), the displacements in terms of $\phi_1(\zeta_1)$ and $\phi_2(\zeta_2)$ are

$$\begin{aligned} u &= 2Re[p_1\phi_1(\zeta_1)+p_2\phi_2(\zeta_2)] \\ v &= 2Re[q_1\phi_1(\zeta_1)+q_2\phi_2(\zeta_2)] \end{aligned} \qquad (20)$$

From Eqs. (12) and (17), the resultant-forces in terms of $\phi_1(\zeta_1)$ and $\phi_2(\zeta_2)$ are

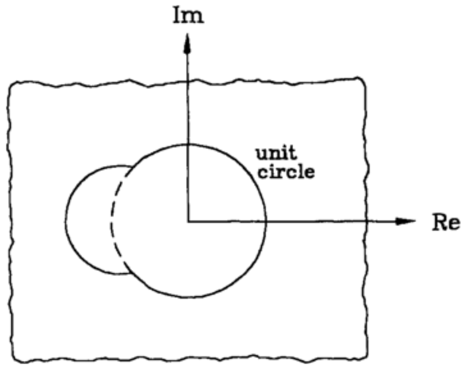

Fig. 2 ζ-transformed plane

$$\begin{aligned} f_1(s)+if_2(s) &= (1+is_1)\phi_1(\zeta_1)+(1+is_2)\cdot \\ &\quad \phi_2(\zeta_2)+(1+i\overline{s_1})\overline{\phi_1(\zeta_1)} \\ &\quad +(1+i\overline{s_2})\overline{\phi_2(\zeta_2)}+c \end{aligned} \qquad (21)$$

Let $S_{\zeta_1}^{+}$ and $S_{\zeta_2}^{+}$ denote the two parameter regions corresponding to ζ_1 and ζ_2, respectively. Their union, $S_{\zeta_1}^{+}$ and $S_{\zeta_2}^{+}$, will be denoted by S_{ζ}^{+}. Figure 2 shows the transformed parameter region S_{ζ}^{+}.

We introduce the following relation :

$$\phi_2(\zeta_2)=B\overline{\phi_1}\left(\frac{1}{\zeta_2}\right)+C\phi_1(\zeta_2) \qquad (22)$$

where

$$\overline{\phi_1}\left(\frac{1}{\zeta}\right)=\overline{\phi_1\left(\frac{1}{\overline{\zeta}}\right)} \qquad (23)$$

$$\begin{aligned} B &= (\overline{s_2}-\overline{s_1})/(s_2-\overline{s_2}), \\ C &= (\overline{s_2}-s_1)/(s_2-\overline{s_2}) \end{aligned} \qquad (24)$$

Traction-free boundary condition on the crack can be ensured if $\phi_1(\zeta)$ is analytic in the region S_{ζ}^{+} and its inversion with respect to the unit circle (Bowie and Freese, 1972). If we assume that the total resultant-forces per unit thickness exerted on the hole boundary are zero, we can express ϕ_1 as follows :

$$\phi_1(\zeta)=\sum_{-\infty}^{\infty}A_n\zeta^n+\sum_{n=1}^{\infty}B_n(\zeta+1)^{-n} \qquad (25)$$

where A_n and B_n are complex constants. Finite dimensional problem can be solved by using the stress function defined in Eq. (25).

The problem can be simplified to selecting the unknowns A_n and B_n in Eq. (25) so that the external boundary conditions may be satisfied. $A_n s$ are directly obtained by applying the boundary conditions at infinity given as

$$\sigma_x=0,\ \sigma_y=T,\ \tau_{xy}=0 \qquad (26)$$

Biaxial problem can be solved by giving σ_x non-zero value. Introducing the condition that stress components σ_x, σ_y, and τ_{xy} remain bounded at infinity and considering that the total resultant-forces per unit thickness exerted on the hole boundary are zero, the structures of the stress functions for large z can be expressed as (Sih, 1973)

$$\phi_1(z_1)\rightarrow Dz_1,\ \phi_2(z_2)\rightarrow (E+iF)z_2 \qquad (27)$$

Substituting Eqs. (26) and (27) into Eq. (8), we obtain

$$\begin{aligned}&(s_1^2+\bar{s}_1^{\,2})D+(s_2^2+\bar{s}_2^{\,2})E+i(s_2^2-\bar{s}_2^{\,2})F=0\\&2D+2E=T\\&(s_1+\bar{s}_1)D+(s_2+\bar{s}_2)E+i(s_2-\bar{s}_2)F=0\end{aligned} \qquad (28)$$

Solving the above equation, we obtain

$$\begin{aligned}&D=(\alpha_2^2+\beta_2^2)T/\Delta\\&E=(\alpha_1^2-\beta_1^2-2\alpha_1\alpha_2)T/\Delta\\&F=\{\alpha_2(\alpha_1^2-\beta_1^2)-\alpha_1(\alpha_2^2-\beta_2^2)\}T/\beta_2\Delta\end{aligned} \qquad (29)$$

where

$$\Delta=2\{(\alpha_2-\alpha_1)^2+(\beta_2^2-\beta_1^2)\} \qquad (30)$$

Considering Eqs (15) and (27), the stress functions $\phi_1(\zeta_1)$ and $\phi_2(\zeta_2)$ for large ζ_1 and ζ_2 can be expressed as

$$\phi_1(\zeta_1)\rightarrow (LD/4)\zeta_1,\ \phi_2(\zeta_2)\rightarrow \{L(E+iF)/4\}\zeta_2 \qquad (31)$$

Taking into account of Eq. (22) and comparing Eq. (25) with Eq. (31), we obtain

$$\begin{aligned}&A_{-1}=L(E-iF-D\bar{C})/4\bar{B}\\&A_1=LD/4\\&A_n=0 \text{ for } n\geq 2 \text{ or } n\leq -2\end{aligned} \qquad (32)$$

Because A_0 is related to translational motion and does not affect the stress field, A_0 can be set to zero.

Thus, the stress function $\phi_1(\zeta)$ in Eq. (25) may be expressed as

$$\phi_1(\zeta)=(LD/4)\zeta+(L/4\bar{B})(E-iF-D\bar{C})\frac{1}{\zeta}+\sum_{n=1}^{\infty}(a_n+ib_n)(\zeta+1)^{-n} \qquad (33)$$

where a_n and b_n are real numbers to be determined. The problem can be simplified to selecting the unknowns a_n and b_n so that the boundary conditions on the circular hole may be satisfied. However, the complex parameters of cross-ply laminates are pure imaginary and loading is symmetric. Thus, we can set $b_n=0$ for all n in Eq. (33). To accomplish numerical analysis, we have to truncate the terms of Eq. (33) over $n=N$.

Substituting Eq. (33) into Eq. (19), the stress are

$$\sigma_x=2Re\left[\frac{s_1^2}{\omega'(\zeta_1)}\left(A_1-\frac{A_{-1}}{\zeta_1^2}\right)+\frac{s_2^2}{\omega'(\zeta_2)}\left\{A_1\left(C-\frac{B}{\zeta_2^2}\right)+\left(\overline{A_{-1}}B-\frac{A_{-1}C}{\zeta_2^2}\right)\right\},+\sum_{n=1}^{N}S_na_n\right],\ \text{etc} \qquad (34)$$

where

$$S_n=\frac{s_1^2}{\omega'(\zeta_1)}(-n)(\zeta_1+1)^{-n-1}+\frac{s_2^2}{\omega'(\zeta_2)}\left\{nB\left(\frac{1}{\zeta_2}+1\right)^{-n-1}\frac{1}{\zeta_2^2}-nC(\zeta_2+1)^{-n-1}\right\} \qquad (35)$$

Substituting Eq. (33) into (20), the displacements are

$$u=2Re\left[p_1\left(A_1\zeta_1+\frac{A_{-1}}{\zeta_1}\right)+p_2\left\{A_1\left(\frac{B}{\zeta_2}+C\zeta_2\right)+\overline{A_{-1}}B\zeta_2+\frac{A_{-1}C}{\zeta_2}\right\}+\sum_{n=1}^{N}D_na_n\right],\ etc \qquad (36)$$

where

$$D_n=p_1(\zeta_1+1)^{-n}+p_2\left\{B\left(\frac{1}{\zeta_2}+1\right)^{-n}+C(\zeta_2+1)^{-n}\right\} \qquad (37)$$

Substituting Eq. (33) into Eq. (21), the resultant-forces are

$$f_1=2Re\left[\left(A_1\zeta_1+\frac{A_{-1}}{\zeta_1}\right)+\left\{A_1\left(\frac{B}{\zeta_2}+C\zeta_2\right)+\overline{A_{-1}}B\zeta_2+\frac{A_{-1}C}{\zeta_2}\right\}+\sum_{n=1}^{N}F_na_n,\ etc \qquad (38)$$

where

$$F_n=(\zeta_1+1)^{-n}+\left\{B\left(\frac{1}{\zeta_2}+1\right)^{-n}+C(\zeta_2+1)^{-n}\right\} \qquad (39)$$

Truncating the unknown terms a_n in Eq. (33) so that the boundary conditions on the circular hole are satisfied with a sufficient accuracy, the stress function may be determined. The stress intensity factors may be evaluated directly from the stress functions $\phi_1(z_1)$ or $\phi_2(z_2)$. In the limit as z_j approaches the crack tip, say $z_0(=L)$, we can express the relation between the stress intensity factors and the stress functions as follows(

Sih and Liebowitz, 1968) :

$$K_I+\frac{K_{II}}{s_2}=2\sqrt{2\pi}\left[\frac{s_2-s_1}{s_2}\right]\lim_{z_1\to z_0}\sqrt{z_1-z_0}\,\phi'_1(z_1) \quad (40)$$

Considering mapping function $z=\omega(\zeta)$ and employing Eqs. (15) ~ (18), we obtain

$$K_I+\frac{K_{II}}{s_2}=2\sqrt{2\pi/L}\left[\frac{s_2-s_1}{s_2}\right]\phi'_1(1) \quad (41)$$

Substituting Eq. (33) into eq. (41), the stress intensity factors can be expressed in terms of coefficients stress function.

$$K_I+\frac{K_{II}}{s_2}=2\sqrt{2\pi/L}\left[\frac{s_2-s_1}{s_2}\right]\cdot \left[A_1-A_{-1}+\sum_{n=1}^{N}\frac{-n}{2^{n+1}}a_n\right] \quad (42)$$

Thus, we can evaluate the stress intensity factors if the coefficients of the stress functions are determined.

4. Numerical Results

The correction factors for a radial crack in cross-ply laminates were calculated by Fortran program based on the foregoing analysis. The stress intensity factors are presented as functions of the normalized crack length $(L-R)/L$ for various types of cross-ply laminates. It is sufficient to apply the collocation argument over a half of the circular hole boundary in this study.

Figure 3 shows the convergence curve for a radial crack in [0/90]s laminate under uniaxial tension. It was found truncation over $n=40$ is sufficient. In this study, stress intensity factors were normalized with those for the case of a central crack of length $2L$ in infinite plate. Material properties of E-glass/epoxy used in the current analysis are as follows :

$E_1=53.74$ GPa$(7.80\times10^6$ psi),
$E_2=17.91$ GPa$(2.60\times10^6$ psi),
$G_{12}=8.96$ GPa$(1.30\times10^6$ psi), $\nu_{12}=0.25$

Figure 4 compares the present results for the case of isotropic infinite plate under uniaxial tension with those of references (Bowie, 1956; Tweed and Rooke, 1976). The isotropic solution was obtained by setting the complex parameters $s_1=1.0\,i$ and $s_2=0.995i$. Bowie(1956) obtained the solution by the boundary collocation method. Tweed and Rooke(1976) obtained the solution by Mellin transform technique. The present results almost coincide with those of references

Figure 5 shows the correction factors for a radial crack in cross-ply laminates under uniaxial tension. The stress intensity factors increase as the ratio $(L-R)/L$ increases up to about 0.2 or 0.3, and then decrease as the ratio increases further. It seems that the stress concentration is greatly dependent on the percentage of 0° plies in cross-ply laminats with a circular hole until the ratio is about 0.2 to 0.3. In the range of small crack length, the stress intensity factor for a radial crack in cross-ply laminates under uniaxial tension becomes larger as the percentage of 0° plies increases. When the crack is short, the stress

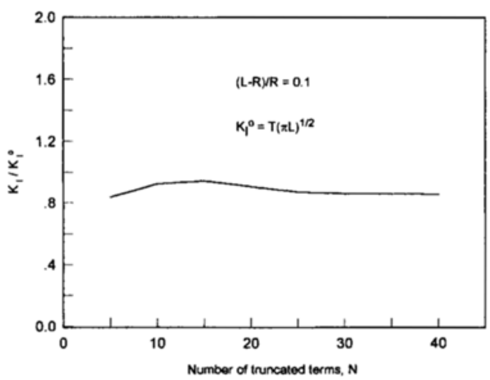

Fig. 3 Convergence curve for a radial crack in $[0/90]_s$ laminate under uniaxial tension

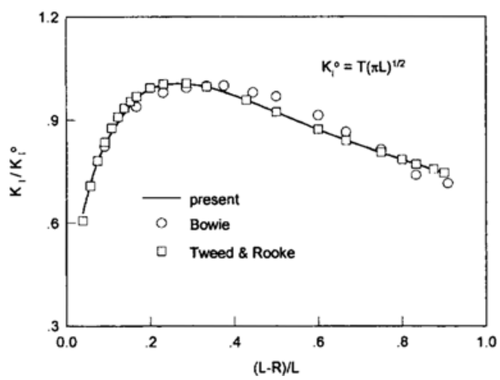

Fig. 4 Correction factors for a radial crack in an isotropic infinite plate under uniaxial tension

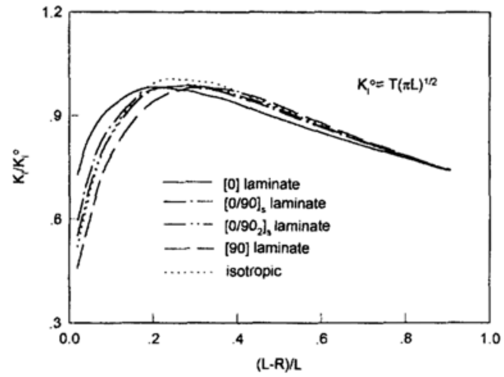

Fig. 5 Correction factors for a radial crack in cross-ply laminates under uniaxial tension

intensity factor for [0] laminate is about fifty percent higher than that for [90] laminate. In the range of large crack length, the stress intensity factors have almost same values regardless of the percentage of 0° plies.

5. Conclusions

Analyzing the problem of a radial crack in cross-ply laminates under uniaxial tension, conclusive remarks can be summarized as follows:

(1) The present results for the case of isotropic infinite plate under uniaxial tension coincide with those of references.

(2) The stress intensity factors for a radial crack in cross-ply laminates under uniaxial tension increase as the ratio $(L-R)/L$ increases up to about 0.2 or 0.3, and then decrease as the ratio increases further.

(3) In the range of small crack length, the stress intensity factor for a radial crack in cross-ply laminates under uniaxial tension becomes larger as the percentage of 0° plies increases.

(4) In the range of large crack length the stress intensity factors for a radial crack in cross-ply laminates under uniaxial tension are insensitive to the percentage of 0° plies.

(5) The formulation developed in this paper can be extended to biaxial problem and finite dimensional problem.

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